PHYS 307 HW PLANE PENDULUM 2023

1) WRITING OUT THE DETAILS BETWEEN TM5'S EQUATIONS 4.28 AND 4.29.

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from which

$$\tau = 4\sqrt{\frac{l}{g}}\int_0^1 [(1-z^2)(1-k^2z^2)]^{-1/2}dz \tag{4.28}$$

Numerical values for integrals of this type can be found in various tables.

For oscillatory motion to result, $|\theta_0| < \pi$, or, equivalently, $\sin(\theta_0/2) = k$, where $-1 < k < +1$. For this case, we can evaluate the integral in Equation 4.28 by expanding $(1-k^2z^2)^{-1/2}$ in a power series:

$$(1-k^2z^2)^{-1/2} = 1 + \frac{k^2z^2}{2} + \frac{3k^4z^4}{8} + \cdots \tag{4.28b}$$

Then, the expression for the period becomes

$$\tau = 4\sqrt{\frac{l}{g}}\int_0^1 \frac{dz}{(1-z^2)^{1/2}}\left[1 + \frac{k^2z^2}{2} + \frac{3k^4z^4}{8} + \cdots\right] \tag{4.28c}$$

$$= 4\sqrt{\frac{l}{g}}\left[\frac{\pi}{2} + \frac{k^2}{2}\cdot\frac{1}{2}\cdot\frac{\pi}{2} + \frac{3k^4}{8}\cdot\frac{3}{8}\cdot\frac{\pi}{2} + \cdots\right] \tag{4.28d}$$

$$= 2\pi\sqrt{\frac{l}{g}}\left[1 + \frac{k^2}{4} + \frac{9k^4}{64} + \cdots\right] \tag{4.28e}$$

If $|k|$ is large (i.e., near 1), then we need many terms to produce a reasonably accurate result. But for small k, the expansion converges rapidly. And because $k = \sin(\theta_0/2)$, then $k \cong (\theta_0/2) - (\theta_0^3/48)$; the result, correct to the fourth order, is

$$\tau \cong 2\pi\sqrt{\frac{l}{g}}\left[1 + \frac{1}{16}\theta_0^2 + \frac{11}{3072}\theta_0^4\right] \tag{4.29}$$

4.28 IS

$$\tau = 4\sqrt{\frac{l}{g}}\int_0^1 \left[(1-z^2)(1-k^2z^2)\right]^{-\frac{1}{2}} dz$$

a) EXPAND THE 2ND BINOMIAL

$$(1 \pm x)^{-1/2} = 1 \mp \frac{x}{2} + \frac{3}{8}x^2 \mp \frac{5}{16}x^3 + \ldots$$

$$(4.28b) \quad \Rightarrow (1-k^2z^2)^{-1/2} = 1 + \frac{k^2z^2}{2} + \frac{3k^4z^4}{8} + \ldots$$

b) SUBSTITUTE INTO 4.28

$$\tau = 4\sqrt{\frac{l}{g}}\int_0^1 \frac{dz}{\sqrt{1-z^2}}\left[1 + \frac{k^2z^2}{2} + \frac{3k^4z^4}{8} + \ldots\right] \qquad (4.28c)$$

WHICH IS A SUM OF ELLIPTICAL INTEGRALS

$$\tau = 4\sqrt{\frac{l}{g}}\left\{\int_0^1 \frac{dz}{\sqrt{1-z^2}} + \frac{k^2}{2}\int_0^1 \frac{z^2dz}{\sqrt{1-z^2}} + \frac{3k^4}{8}\int_0^1 \frac{z^4dz}{\sqrt{1-z^2}}\right\}$$

c) APPENDIX E.3 GIVES SOLUTIONS OF THESE USING Γ FUNCTIONS

$$\int_0^1 x^m(1-x^2)^n dx = \frac{\Gamma(n+1)\,\Gamma\left(\frac{m+1}{2}\right)}{2\Gamma\left(n+\frac{m+3}{2}\right)} \qquad (E.27a)$$

$$\Rightarrow \int_0^1 \frac{dz}{\sqrt{1-z^2}},\ m=0,\ n=-\tfrac{1}{2} \text{ GIVES } \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{2\Gamma(-\frac{1}{2}+\frac{3}{2})} = \frac{[\Gamma(\frac{1}{2})]^2}{2\Gamma(1)}$$

$$\boxed{\int_0^1 \frac{dz}{\sqrt{1-z^2}} = \frac{\pi}{2}} \qquad \Gamma(\tfrac{1}{2}) = \sqrt{\pi},\ \Gamma(1)=1,\ \Gamma(n+1)=n\Gamma(n),\ \Gamma(2)=1$$

$$\int_0^1 \frac{z^2dz}{\sqrt{1-z^2}},\ m=2,\ n=-\tfrac{1}{2} \text{ GIVES } \frac{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})}{2\Gamma(-\frac{1}{2}+\frac{5}{2})} = \frac{\Gamma(\frac{1}{2})\left[\frac{1}{2}\Gamma(\frac{1}{2})\right]}{2\Gamma(2)}$$

$$\boxed{\int_0^1 \frac{z^2dz}{\sqrt{1-z^2}} = \frac{\sqrt{\pi}\left[\frac{1}{2}\sqrt{\pi}\right]}{2} = \frac{\pi}{4}}$$

$\longrightarrow$

1) CONTINUED

$$\int_0^1 \frac{z^4\,dz}{\sqrt{1-z^2}}, \quad m=4,\ n=-\tfrac{1}{2} \text{ GIVES } \frac{\Gamma(\frac{1}{2})\Gamma(\frac{5}{2})}{2\Gamma(-\frac{1}{2}+\frac{7}{2})} = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{5}{2})}{2\Gamma(3)}$$

$$\Gamma\left(\tfrac{5}{2}\right) = \tfrac{3}{2}\Gamma\left(\tfrac{3}{2}\right) = \tfrac{3}{2}\left[\tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right)\right] = \tfrac{3}{4}\Gamma\left(\tfrac{1}{2}\right)$$

$$\Gamma(3) = 2\Gamma(2) = 2$$

$$\boxed{\int_0^1 \frac{z^4\,dz}{\sqrt{1-z^2}} = \frac{\sqrt{\pi}\left[\frac{3}{4}\sqrt{\pi}\right]}{2(2)} = \frac{3\pi}{16}}$$

SUBSTITUTING THIS GIVES

$$T = 4\sqrt{\frac{l}{g}}\left\{\frac{\pi}{2} + \frac{\pi k^2}{8} + \frac{9\pi k^4}{128}\right\}$$

$$T = 2\pi\sqrt{\frac{l}{g}}\left\{1 + \frac{k^2}{4} + \frac{9k^4}{64}\right\} \qquad (4.28c)$$

d) EXPAND $k = \sin\left(\frac{\theta_0}{2}\right)$ USING $\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

$$k \approx \frac{\theta_0}{2} - \frac{1}{6}\left(\frac{\theta_0^3}{8}\right) + \frac{1}{120}\left(\frac{\theta_0^5}{32}\right) - \ldots$$

KEEPING TERMS WITH θ^n WHERE $n \le 4$

$$k \approx \frac{\theta_0}{2} - \frac{\theta_0^3}{48}$$

GIVING

$$k^2 \approx \left(\frac{\theta_0}{2} - \frac{\theta_0^3}{48}\right)^2 = \frac{\theta_0^2}{4} - \cancel{2}\left(\frac{\theta_0}{\cancel{2}}\right)\left(\frac{\theta_0^3}{48}\right) + \cancel{\frac{\theta_0^6}{(48)^2}}$$

NEGLECT $(n > 4)$

$$\left.\begin{aligned} k^2 &\approx \frac{\theta_0^2}{4} - \frac{\theta_0^4}{48} \\ \text{AND} \quad k^4 &\approx \frac{\theta_0^4}{16} \end{aligned}\right\} \text{KEEPING } \theta^n \text{ WITH } n \le 4$$

SUBSTITUTE BACK INTO (4.28c)

$$T = 2\pi\sqrt{\frac{l}{g}}\left\{1 + \frac{1}{4}\left[\frac{\theta_0^2}{4} - \frac{\theta_0^4}{48}\right] + \frac{9}{64}\left[\frac{\theta_0^4}{16}\right]\right\}$$

1) CONTINUED MORE

$$\tau = 2\pi\sqrt{\frac{l}{g}}\left\{1 + \frac{\theta_0^2}{16} \underbrace{- \frac{\theta_0^4}{192} + \frac{9\theta_0^4}{1024}}_{\text{FIND COMMON DENOMINATOR}}\right\}$$

$192 = 4 \times 48 = 4 \times 8 \times 6 = 2^2 2^3 2 \cdot 3 = 2^6 \cdot 3$

$1024 = 16 \times 64 = 2 \times 8 \times 8 \times 8 = 2^{10}$

$\Rightarrow 2^{10} \times 3 = (3)(1024) = 3072$

$$\tau = 2\pi\sqrt{\frac{l}{g}}\left\{1 + \frac{\theta_0^2}{16} + \frac{[-16 + 27]\,\theta_0^4}{3072}\right\}$$

$$\boxed{\tau = 2\pi\sqrt{\frac{l}{g}}\left\{1 + \frac{\theta_0^2}{16} + \frac{11\,\theta_0^4}{3072}\right\} \qquad (4.29)}$$

QED!